The infinity Quillen functor, Maurer-Cartan elements and DGL realizations

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Abstract

We prove that for any complete differential graded Lie algebra (cDGL) L, its geometrical realization $\langle L\rangle_\bullet = \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}_\bullet, L)$ via the cosimplicial free cDGL $\mathfrak{L}_\bullet = \widehat{\mathbb{L}}(s^{-1}\Delta^\bullet)$ is homotopy equivalent to the classical Hinich realization $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$. For it, we need to detect certain cDGL morphisms as Maurer-Cartan elements of corresponding L_∞-algebra structures.

Introduction

In [3] we introduced a cosimplicial complete differential graded Lie algebra (cDGL henceforth) $\mathfrak{L}_\bullet$ by extending the Lawrence-Sullivan interval [18] to any simplex: for each $n \geq 1$, $\mathfrak{L}_n = \widehat{\mathbb{L}}(s^{-1}\Delta^n)$ is the free cDGL in which $s^{-1}\Delta^n$ together with the linear part of the differential is the (desuspension) of the chain complex of the standard n-simplex, and the vertices correspond to Maurer-Cartan elements. This let us geometrically realize any cDGL L as the simplicial set

$$\langle L\rangle_\bullet = \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}_\bullet, L).$$

We also proved [3, §8] that under usual bounding and finite type assumptions this simplicial set is homotopy equivalent to any other known realization

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of a cDGL L, i.e., the λ-Quillen functor on L [21], the realization of the Chevalley-Eilenberg cochain functor on L [2], and the Hinich "contents or nerve" of L [12]. The latter, carefully studied in [1, 9], is defined as the simplicial set of Maurer-Cartan elements of the simplicial DGL $\mathscr{A}_\bullet \otimes L$ in which $\mathscr{A}_\bullet$ denotes the simplicial commutative differential graded algebra of PL-differential forms on the standard simplices [6, 25]. In this paper, we show that, with full generality, this is homotopy equivalent to our realization:

Theorem 0.1. *For any* cDGL L *there are explicit homotopy equivalences*

$$\mathrm{MC}(\mathscr{A}_\bullet \otimes L) \underset{i}{\overset{p}{\rightleftarrows}} \langle L \rangle_\bullet$$

which make $\langle L \rangle_\bullet$ *a strong homotopy retract of* $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$.

One of the key ingredients is adding two new entries to the so called "Rosetta Stone" of higher structures [19, §10.1.9]: any homotopy retract (see next section for precise definitions)

$$\circlearrowright C \rightleftarrows V$$

of the cocommutative differential graded coalgebra (CDGC) C produces a free cDGL $\mathfrak{L}(V) = \mathbb{L}(s^{-1}V)$ and a homotopy retract

$$\circlearrowright \mathrm{Hom}(C, L) \rightleftarrows \mathrm{Hom}(V, L)$$

of the convolution Lie algebra $\mathrm{Hom}(C, L)$ for any DGL L. This induces an L_∞-algebra structure on $\mathrm{Hom}(V, L)$ for which the following holds (see Theorem 3.2).

Theorem 0.2. *For any* cDGL L*, there is a bijection*

$$\mathrm{MC}\,\mathrm{Hom}(V, L) \cong \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V), L).$$

proved

The dual result is also attained although finite type requirements are essential: a homotopy retract as above, in which now C is a commutative differential graded algebra and V is of finite type, also provides a free cDGL $\mathfrak{L}(V^\sharp)$ and a homotopy retract

$$\circlearrowright C \otimes L \rightleftarrows V \otimes L$$

of the Lie algebra $C \otimes L$ for any DGL L. This induces an L_∞-algebra structure on $V \otimes L$ which is naturally isomorphic to $\mathrm{Hom}(V^\sharp, L)$. Then (see Theorem 3.5):

Theorem 0.3. *For any* cDGL L *there is a bijection*

$$\mathrm{MC}(V \otimes L) \cong \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V^\sharp), L).$$

These results can be extended to the general case in which C of the homotopy retract above is either a C_∞-algebra or C_∞-coalgebra. Nevertheless, for our purposes we only need them as stated.

For some of the above statements we need Theorem 2.1, consisting, in the spirit of [22, §1], of new Lie bracket formulae for Lie polynomials on a general tensor algebra.

Next section contains notation and recalls explicit descriptions of transferred structures. In Section 2 we prove Theorem 2.1 and Section 3 contains Theorem 0.2 and 0.3. Finally, Theorem 0.1 is proved in Section 4.

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1 Preliminaries

With the aim of fixing notation, we begin by recalling known facts and setting some generic assumptions on the algebraic structures of our concern. Abusing notation, we will not distinguish a given category $\mathcal{C}$ from the class of its objects. Its morphism sets are denoted by $\mathrm{Hom}_{\mathcal{C}}$ and unadorned Hom denotes just linear maps. The coefficient field for any algebraic object $\mathbb{K}$ is assumed to be of characteristic zero. Any graded object is considered $\mathbb{Z}$-graded and under no finite type assumptions unless explicitly specified otherwise. The degree of a homogeneous element x in such an object will be denoted by $|x|$.

An A_∞-*algebra* is a graded vector space A endowed with a family of linear maps,

$$m_k\colon A^{\otimes k} \longrightarrow A, \quad , k \geq 1,$$

of degree $k-2$, such that for all $i \geq 1$,

$$\sum_{k=1}^{i} \sum_{n=0}^{i-k} (-1)^{k+n+kn} m_{i-k+1}(\mathrm{id}^n \otimes m_k \otimes \mathrm{id}^{\otimes i-k-n}) = 0.$$

An A_∞-algebra A is *commutative*, or it is a C_∞-*algebra* if, for each $k \geq 2$, the k-th multiplication m_k vanishes on the *shuffle products*, that is, $m_k \nu_k = 0$,

with

$$\nu_k \colon A^{\otimes k} \to A^{\otimes k}, \quad \nu_k(a_1 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k} \sum_{\sigma \in S(i,k-i)} \varepsilon_\sigma \, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},$$

where $S(i, k-i)$ denotes the set of $(i, k-i)$-shuffles, i.e., permutations σ such that $\sigma^{-1}(1) < \cdots < \sigma^{-1}(i)$ and $\sigma^{-1}(i+1) < \cdots < \sigma^{-1}(k)$.

A *differential graded algebra*, DGA henceforth (CDGA if it is commutative), is an A_∞-algebra for which $m_k = 0$ for all $k \geq 3$. In this case $m_1 = d$ and $m_2 = m$ are the *differential* and *multiplication* respectively.

Dually, an A_∞*-coalgebra* is a graded vector space C endowed with a family of linear maps,

$$\Delta_k \colon C \longrightarrow C^{\otimes k}, \quad , k \geq 1,$$

of degree $k-2$, such that for all $i \geq 1$,

$$\sum_{k=1}^{i} \sum_{n=0}^{i-k} (-1)^{k+n+kn} (\mathrm{id}^{\otimes i-k-n} \otimes \Delta_k \otimes \mathrm{id}^n) \Delta_{i-k+1} = 0.$$

An A_∞-coalgebra C is *cocommutative*, or it is a C_∞*-coalgebra* if, for each $k \geq 2$, the *unshuffle products* vanish on the image of the k-th comultiplication Δ_k, that is, $\tau \circ \Delta_k = 0$ with

$$\tau \colon C^{\otimes k} \to C^{\otimes k}, \quad \tau(c_1 \otimes \cdots \otimes c_k) = \sum_{i=1}^{k} \sum_{\sigma \in S(i,k-i)} \varepsilon_\sigma \, c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(k)}.$$

A *differential graded coalgebra*, DGC henceforth (CDGC if it is commutative), is an A_∞-coalgebra for which $\Delta_k = 0$ for all $k \geq 3$. In this case $\Delta_1 = \delta$ and $\Delta_2 = \Delta$ are the *codifferential* and *comultiplication* respectively.

An L_∞*-algebra* is a graded vector space L together with linear maps $\ell_k \colon L^{\otimes k} \to L$ of degree $k-2$, for $k \geq 1$, satisfying:

(i) For any permutation σ of n elements, and any n-tuple $x_1, \ldots, x_n$ of homogeneous elements of L,

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \varepsilon_\sigma \varepsilon \ell_k(x_1, \ldots, x_n),$$

where ε_σ is the signature of the permutation and ε is the sign given by the Koszul convention.

(ii) For any $n \geq 1$ and any n-tuple $x_1, \dots, x_n$ of homogeneous elements of L, the *generalized Jacobi identity* holds,

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i,n-i)} \varepsilon_\sigma \varepsilon (-1)^{i(j-1)} \ell_{n-i}(\ell_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(n)}) = 0,$$

The set of *Maurer-Cartan elements* of an L_∞-algebra L is defined as

$$\mathrm{MC}(L) = \{x \in L_{-1} \mid \sum_{k \geq 0} \frac{1}{k!} \ell_k(x, \dots, x) = 0\}.$$

A *differential graded Lie algebra*, DGL henceforth, is an L_∞-algebra L for which $\ell_k = 0$ for all $k \geq 3$. In this case $\ell_1 = \partial$ and $\ell_2 = [\ ,\ ]$ are the *differential* and the *Lie bracket* respectively. In this case $\mathrm{MC}(L) = \{x \in L_{-1} \mid \partial x = -\frac{1}{2}[x, x]\}$. A DGL L is called *free* if it is free as a Lie algebra, that is, $L = \mathbb{L}(V)$ for some graded vector space V. Recall that $\mathbb{L}(V) \subset T(V)$ is the Lie algebra generated by commutators on V. Given L a DGL, its *completion* $\widehat{L}$ is the limit

$$\widehat{L} = \varprojlim_n L/L^n$$

where $L^1 = L$ and for $n \geq 2$, $L^n = [L, L^{n-1}]$. A *complete differential graded Lie algebra*, cDGL henceforth, is a DGL isomorphic to its completion.

Consider a diagram

$$K \circlearrowright M \underset{i}{\overset{p}{\rightleftarrows}} V$$

where i and p are chain maps for which $pi = \mathrm{id}_V$ and $ip \simeq \mathrm{id}_M$ through the chain homotopy K. We encode this data as (M, V, i, p, K) and call it a *homotopy retract*.

For such a retract, the *homotopy transfer theorem* [8, 15, 16, 19, 20], also known as the *homological perturbation lemma* [10, 11, 13, 14] permits to transfer any additional structure on M to the corresponding infinity version on V:

Theorem 1.1. *Given (M, V, i, p, K) a homotopy retract in which M is either a DGA (resp. CDGA), DGC (resp. CDGC) or DGL, there exists an A_∞-algebra (resp. C_∞-algebra), A_∞-coalgebra (resp. C_∞-coalgebra) or L_∞-algebra structure in V, unique up to isomorphism, and quasi-isomorphisms in the corresponding infinity structure*

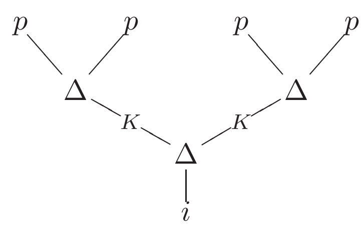
$$M \underset{I}{\overset{P}{\rightleftarrows}} V$$

extending i and p and such that $PI = \mathrm{id}_V$.

The general fact that CDGA's transfer to C_∞-algebras is proved in [5, Theorem 12]. A dual argument proves the analogue for commutative DGC's.

As we shall strongly use it, we describe in each case, the explicit description of the transferred structure in the above theorem. In what follows, for any $k \geq 2$, $\mathscr{P}\mathscr{T}_k$ denote the set of isomorphism classes of planar rooted binary trees of k leaves, while $\mathscr{T}_k$ consists of isomorphism classes of (non planar) rooted binary trees with k leaves.

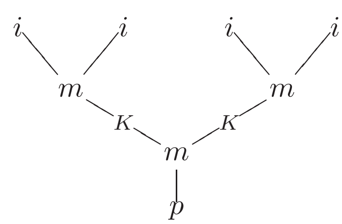
Let first $M = (C, \delta, \Delta)$ be a (commutative) DGC. For each $T \in \mathscr{P}\mathscr{T}_k$, we define a linear map $\Delta_T\colon V \to V^{\otimes k}$ as follows: label the root by i, each internal edge by K, each internal vertex by Δ, and each leaf by p. Then, Δ_T is defined as the composition of the different labels moving up from the root to the leaves. For instance, the tree $T \in \mathscr{P}\mathscr{T}_4$



yields the map $\Delta_T = p^{\otimes 4} \circ (\Delta \circ K \otimes \Delta \circ K) \circ \Delta \circ i\colon V \to V^{\otimes 4}$. By Theorem 1.1 the transferred (commutative) A_∞-coalgebra structure in V is given by $\{\Delta_k\}_{k \geq 1}$, where $\Delta_1 = d$ and, for $k \geq 2$,

$$\Delta_k = \sum_{T \in \mathscr{P}\mathscr{T}_k} \Delta_T. \tag{1}$$

Dually, if $M = (A, d, m)$ is a (commutative) DGA (m denotes its multiplication), for each $T \in \mathscr{P}\mathscr{T}_k$, we define a linear map $m_T\colon V^{\otimes k} \to V$ as follows: label the root by p, each internal edge by K, each internal vertex by m, and each leaf by i. Then, m_T is defined as the composition of the different labels moving down from the leaves to the root. Now, the same tree above

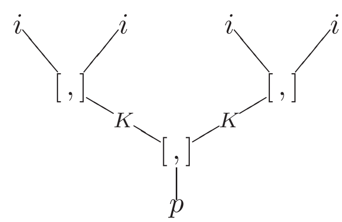


produces $m_T = p \circ m \circ (K \circ m \otimes K \circ m) \circ i^{\otimes 4}\colon V^{\otimes 4} \to V$. The transferred (commutative) A_∞-algebra structure in V provided by Theorem 1.1 is given

by $\{m_k\}_{k\geq 1}$, where $m_1 = d$ and, for $k \geq 2$,

$$m_k = \sum_{T \in \mathscr{P}\mathscr{T}_k} m_T. \tag{2}$$

The case in which $M = (L, \partial, [\ ,\ ])$ is a DGL is slightly different. For each T in $\mathscr{T}_k$ define a linear map $\ell_T\colon V^{\otimes k} \longrightarrow V$ as follows: choose a planar embedding of T, label the root by p, each internal edge by K, each internal vertex by $[\ ,\ ]$, and each leaf by i. Then, the map $\widetilde{\ell}_T\colon V^{\otimes k} \longrightarrow V$ is defined as the composition of the different labels moving down from the leaves to the root. For instance the planar embedding



of the corresponding $T \in \mathscr{T}_4$ produces the map

$$p \circ [\,,] \circ (K \circ [\,,] \otimes K \circ [\,,]) \circ i^{\otimes 4}.$$

Define

$$\ell_T = \widetilde{\ell}_T \circ \mathcal{S}_k$$

where

$$\mathcal{S}_k\colon V^{\otimes k} \to V^{\otimes k}, \quad \mathcal{S}_k(a_1 \dots a_k) = \sum_{\sigma \in S_k} \varepsilon_\sigma \varepsilon\, a_{\sigma(1)} \dots a_{\sigma(k)},$$

is the symmetrization map, in which ε_σ denotes the signature of the permutation and ε is the sign given by the Koszul convention. The map ℓ_T is independent of the chosen planar embedding and, by Theorem 1.1, the transferred L_∞-algebra structure in V is given by $\{\ell_k\}_{k\geq 1}$, where $\ell_1 = d$ and, for $k \geq 2$,

$$\ell_k = \sum_{T \in \mathscr{T}_k} \frac{\ell_T}{|\operatorname{Aut} T|} \tag{3}$$

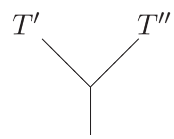
where $\operatorname{Aut} T$ is the automorphism group of the tree T.

2 Lie polynomials

Let $\varphi\colon V \to V \otimes V$ be a linear map whose image is a Lie polynomial, i.e., $\mathrm{Im}\varphi \subset \mathbb{L}^2(V)$ and let $T \in \mathscr{P}\mathscr{T}_k$. Define a linear map

$$\varphi_T\colon V \to V^{\otimes k}$$

recursively as follows $\varphi_T\colon V \to V^{\otimes k}$ by $\varphi_| = \mathrm{id}_V$ for the trivial tree $|$, $\varphi_{\mathsf{Y}} = \varphi$, and if T is of the form



(4)

denoted in the sequel by $T' \mathsf{Y}\, T''$,

$$\varphi_T = \Big(\varphi_{T'} \otimes \varphi_{T''}\Big) \circ \varphi$$

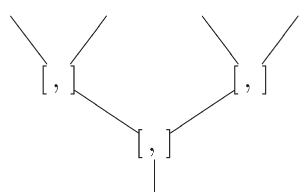
Finally, given $\psi\colon V \to W$ a homogeneous linear map we set

$$P_T\colon V \to W^{\otimes k}, \qquad P_T = \psi^{\otimes k} \circ \varphi_T. \tag{5}$$

We also consider the linear map

$$Q_T\colon W^{\otimes k} \to \mathbb{L}^k(W) \tag{6}$$

defined as the nested commutator bracket moving down from the leaves to the root of the tree T. For example, the tree



produces $Q_T = [\,,\,] \circ ([\,,\,] \otimes [\,,\,])\colon W^{\otimes 4} \to \mathbb{L}^4(W)$.

Now, for each $T \in \mathscr{P}\mathscr{T}_k$ we consider the subset $\overline{T} \subset \mathscr{P}\mathscr{T}_k$ consisting of all the planar embeddings of trees which are isomorphic to T as non planar trees. We prove:

Theorem 2.1. *For each $T \in \mathscr{PT}_k$,*

$$\frac{1}{|\operatorname{Aut} T|} Q_T \circ P_T = \sum_{S \in \overline{T}} P_S.$$

Here $|\operatorname{Aut} T|$ denotes the automorphism of T as a non planar rooted tree.

Proof. We proceed by induction on the number of leaves. Given $v \in V$, write $\varphi(v) = \sum_i a_i \otimes b_i \pm b_i \otimes a_i = \sum_i [a_i, b_i]$. Then,

$$\begin{aligned}
P_{\curlyvee}(v) &= \sum_i [\psi(a_i), \psi(b_i)] = \sum_i \frac{1}{2}\big([\psi(a_i), \psi(b_i)] \pm [\psi(b_i), \psi(a_i)]\big) \\
&= \frac{1}{2} Q_{\curlyvee} \circ P_{\curlyvee}(v)
\end{aligned}$$

which proves the statement for $k = 2$.

For the general case, and to avoid excessive notation, we write $\varphi(v) = ab - (-1)^{|a||b|} ba$ and consider ψ to be id_V. As in (4) decompose any given tree $T \in \mathscr{PT}_k$ in the form $T = T' \curlyvee T''$.

Suppose first that $T' \neq T''$. Then, $\overline{T} = \{S' \curlyvee S'', S'' \curlyvee S'\}_{S' \in \overline{T}', S'' \in \overline{T}''}$ and $|\operatorname{Aut} T'| = |\operatorname{Aut} T'||\operatorname{Aut} T''|$. Therefore,

$$\begin{aligned}
\sum_{S \in \overline{T}} P_S(v) &= \Big(\sum_{S' \in \overline{T}'} P_{S'} \otimes \sum_{S'' \in \overline{T}''} P_{S''} + \sum_{S'' \in \overline{T}''} P_{S''} \otimes \sum_{S' \in \overline{T}'} P_{S'}\Big)(ab - (-1)^{|a||b|} ba) \\
&= \sum_{S' \in \overline{T}'} P_{S'}(a) \otimes \sum_{S'' \in \overline{T}''} P_{S''}(b) - (-1)^{|a||b|} \sum_{S' \in \overline{T}'} P_{S'}(b) \otimes \sum_{S'' \in \overline{T}''} P_{S''}(a) \\
&+ \sum_{S'' \in \overline{T}''} P_{S''}(a) \otimes \sum_{S' \in \overline{T}'} P_{S'}(b) - (-1)^{|a||b|} \sum_{S'' \in \overline{T}''} P_{S''}(b) \otimes \sum_{S' \in \overline{T}'} P_{S'}(a) \\
&= \Big[\sum_{S' \in \overline{T}'} P_{S'}(a), \sum_{S'' \in \overline{T}''} P_{S''}(b)\Big] - (-1)^{|a||b|} \Big[\sum_{S' \in \overline{T}'} P_{S'}(b), \sum_{S'' \in \overline{T}''} P_{S''}(a)\Big] \\
&= \Big[\frac{1}{|\operatorname{Aut} T'|} Q_{T'} \circ P_{T'}(a), \frac{1}{|\operatorname{Aut} T''|} Q_{T''} \circ P_{T''}(b)\Big] \\
&- (-1)^{|a||b|} \Big[\frac{1}{|\operatorname{Aut} T'|} Q_{T'} \circ P_{T'}(b), \frac{1}{|\operatorname{Aut} T''|} Q_{T''} \circ P_{T''}(a)\Big] \\
&= \frac{1}{|\operatorname{Aut} T|} Q_T \circ P_T(v)
\end{aligned}$$

Assume now $T = R \curlyvee R$. In this case $\overline{T} = \{S' \curlyvee S''\}_{S',S''\in\overline{R}}$ and $|\operatorname{Aut} T| = 2|\operatorname{Aut} R|^2$. Then write,

$$\sum_{S\in\overline{T}} P_S(v) = \Big(\sum_{S'\in R} P_{S'} \otimes \sum_{S''\in R} P_{S''}\Big)(ab - (-1)^{|a||b|}ba)$$
$$= \frac{1}{2}\Big(\sum_{S'\in R} P_{S'} \otimes \sum_{S''\in R} P_{S''} + \sum_{S''\in R} P_{S''} \otimes \sum_{S'\in R} P_{S'}\Big)(ab - (-1)^{|a||b|}ba),$$

which coincides with $\frac{1}{|\operatorname{Aut} T|} Q_T \circ P_T(v)$ by the same computation of the previous case. $\square$

3 Maurer-Cartan elements and Lie morphisms

We begin by the following observation: an A_∞-coalgebra structure on a graded vector space V corresponds univocally to a differential in the complete tensor algebra $\widehat{T}(s^{-1}V) = \Pi_{n\geq 0}T^n(s^{-1}V)$ on the desuspension of C, $(s^{-1}V)_p = V_{p+1}$. Indeed, such a differential d is determined by its image on $s^{-1}V$, which is written as a sum $d = \sum_{k\geq 1} d_k$, with $d_k(s^{-1}C) \subset T^k(s^{-1}V)$, for $k \geq 1$. Then, the operators $\{\Delta_k\}_{k\geq 1}$ and $\{d_k\}_{k\geq 1}$ define each other via

$$\begin{aligned} \Delta_k &= -s^{\otimes k} \circ d_k \circ s^{-1} \colon V \to V^{\otimes k}, \\ d_k &= -(-1)^{\frac{k(k-1)}{2}} (s^{-1})^{\otimes k} \circ \Delta_k \circ s \colon s^{-1}V \to T^k(s^{-1}V). \end{aligned} \tag{7}$$

Moreover, from Theorem 2.1 we easily deduce a short proof of a long standing fact whose history is described in [18, Introduction].

Theorem 3.1. *If V is a C_∞-coalgebra, the differential d of any generator $s^{-1}v \in s^{-1}V$ is a Lie polynomial, that is, $ds^{-1}v \in \widehat{\mathbb{L}}(s^{-1}V)$.*

Proof. Simply observe that each Δ_T in the expression of Δ_k in (1) is of the form $P_T \circ i$ where $P_T \colon C \to V^{\otimes k}$ is the map in (5) associated to $\psi = p$ and $\varphi = (K \otimes K) \circ \Delta$, which is a Lie polynomial since C is cocommutative. Apply now Theorem 2.1 and the result follows. $\square$

In particular, the differential d makes $\widehat{\mathbb{L}}(s^{-1}V)$ a cDGL which we call the ∞-*Quillen functor* and denote it by $\mathfrak{L}(V)$.

Now, let C be a CDGC and let L be a DGL. Recall that the linear maps $\operatorname{Hom}(C, L)$ have a natural DGL structure with the usual differential given by the bracket $[\partial, \delta]$ of the differentials in L and C respectively, and the *convolution Lie bracket* $[f, g] = [\,,\,] \circ (f \otimes g) \circ \Delta$.

Hence, if (C, V, i, p, K) is a homotopy retract, then

$$(\mathrm{Hom}(C, L), \mathrm{Hom}(V, L), i^*, p^*, K^*)$$

is again a homotopy retract which, by Theorem 1.1, induces an L_∞-structure on $\mathrm{Hom}(V, L)$.

We now prove that, if L is complete, then the restriction,

$$\mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V), L) \longrightarrow \mathrm{Hom}_{-1}(V, L), \quad f \mapsto f \circ s^{-1},$$

of any cDGL morphism to its generators provides the following bijection.

Theorem 3.2. *For any* cDGL L*,* $\mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V), L) \cong \mathrm{MC}\big(\mathrm{Hom}(V, L)\big)$.

For $k \geq 2$, let Δ_k and ℓ_k denote, as in (1) and (3), the k-th diagonal and k-th bracket induced on V and $\mathrm{Hom}(V, L)$ respectively. A straightforward computation provides the following.

Lemma 3.3. *For each* $T \in \mathscr{P}\mathscr{T}_k$*,*

$$\widetilde{\ell}_T = \gamma \circ Q_T \circ (- \otimes \overset{k}{\cdots} \otimes -) \circ \Delta_T.$$

$\square$

Here Q_T is the map (6) and $\gamma\colon \mathbb{L}^k(L) \to L$ is the Lie bracketing morphism induced by the identity id_L.

Proof of Theorem 3.2. We have to show that $f \in \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V), L)$ if and only if $fs^{-1} \in \mathrm{MC}\big(\mathrm{Hom}(V, L)\big)$. In other words,

$$\partial f(s^{-1}v) + \sum_{k\geq 1} f(d_k s^{-1} v) = 0 \text{ if and only if } \sum_{k\geq 1} \frac{1}{k!} \ell_k(fs^{-1}, \dots, fs^{-1})(v) = 0,$$

for any $v \in V$, being $d = \sum_{k\geq 1} d_k$ as in (7). Recall that $\ell_1(fs^{-1})$ is precisely the differential in $\mathrm{Hom}(V, L)$, i.e., $\ell_1(fs^{-1})(v) = \partial f(s^{-1}v) + f(d_1 s^{-1} v)$, and therefore it suffices to shows that for any $v \in V$,

$$\sum_{k\geq 2} f(d_k s^{-1} v) = 0 \quad \text{if and only if} \quad \sum_{k\geq 2} \frac{1}{k!} \ell_k(fs^{-1}, \dots, fs^{-1})(v) = 0.$$

In fact we prove that, as maps,

$$f d_k s^{-1} = -\frac{1}{k!} \ell_k(fs^{-1}, \dots, fs^{-1}), \quad \text{for any } k \geq 2.$$

On the one hand, using formula (3) first and then Lemma 3.3,

$$\begin{aligned}
\frac{1}{k!}\ell_k(fs^{-1},\dots,fs^{-1}) &= \frac{1}{k!}\sum_{T\in\mathscr{T}_k}\frac{\ell_T(fs^{-1},\dots,fs^{-1})}{|\operatorname{Aut}T|}\\
&= \sum_{T\in\mathscr{T}_k}\frac{\widetilde{\ell}_T(fs^{-1},\dots,fs^{-1})}{|\operatorname{Aut}T|}\\
&= \sum_{T\in\mathscr{T}_k}\frac{\gamma\circ Q_T\circ(fs^{-1})^{\otimes k}\circ\Delta_T}{|\operatorname{Aut}T|}.
\end{aligned}$$

Now observe that $(fs^{-1})^{\otimes k}\circ\Delta_T$ is of the form $P_T\circ i$ where $P_T\colon C\to L^{\otimes k}$ is the map in (5) associated to $\psi = fs^{-1}p$ and $\varphi=(K\otimes K)\circ\Delta$ which is a Lie polynomial since C is cocommutative. Hence, by Theorem 2.1, the above reduces to

$$\sum_{T\in\mathscr{P}\mathscr{T}_k}\gamma\circ(fs^{-1})^{\otimes k}\circ\Delta_T.$$

On the other hand, in view of equation (7), formula (1), an using that f commutes with Lie brackets, we deduce:

$$\begin{aligned}
fd_ks^{-1} &= -(-1)^{\frac{k(k-1)}{2}}f\circ(s^{-1})^{\otimes k}\circ\Delta_k\circ s\circ s^{-1}\\
&= -(-1)^{\frac{k(k-1)}{2}}f\circ(s^{-1})^{\otimes k}\circ\sum_{T\in\mathscr{P}\mathscr{T}_k}\Delta_T\\
&= -\sum_{T\in\mathscr{P}\mathscr{T}_k}\gamma\circ(fs^{-1})^{\otimes k}\circ\Delta_T.
\end{aligned}\tag{8}$$

□

In the dual setting, let A be a CDGA, let L be a DGL and recall that $A\otimes L$ has a natural DGL structure given by $[a\otimes x, b\otimes y]=(-1)^{|x||b|}ab\otimes[x,y]$, $\partial(a\otimes x)=da\otimes x+(-1)^{|a|}a\otimes\partial x$.

Hence, if (A,V,i,p,K) is a homotopy retract,

$$(A\otimes L, V\otimes L, i\otimes\mathrm{id}_L, p\otimes\mathrm{id}_L, K\otimes\mathrm{id}_L)$$

is again a homotopy retract which, by Theorem 1.1, induces an L_∞-structure on $V\otimes L$.

Remark 3.4. To identify its Maurer-Cartan set as cDGL morphisms it is necessary to assume that V is a finite type vector space and we do so henceforth. Under this assumption the map

$$\Psi\colon V\otimes L\longrightarrow\operatorname{Hom}(V^\sharp,L),\quad \Psi(a\otimes x)(\alpha)=(-1)^{|a||x|}\langle a,\alpha\rangle x,$$

is an isomorphism of graded vector spaces. On the other hand, given $k \geq 1$, let m_k denote, as in (2), the kth product in the C_∞-algebra induced on V. Its dual, once composed with the canonical isomorphism $(V^{\otimes k})^\sharp \cong V^{\sharp\otimes k}$ (whose existence requires also V to be of finite type),

$$m_k^\sharp \colon V^\sharp \longrightarrow V^{\sharp\otimes k}$$

defines a C_∞-coalgebra on $V^\sharp$ which, by Theorem 3.1, produces the cDGL $\mathfrak{L}(V^\sharp) = \widehat{\mathbb{L}}(s^{-1}V^\sharp)$.

We show that again, if L is complete, the restriction,

$$\mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V^\sharp), L) \longrightarrow \mathrm{Hom}_{-1}(V^\sharp, L) \cong (V \otimes L)_{-1}$$

of any DGL morphism to its generators provides the following bijection.

Theorem 3.5. *For any* cDGL L, $\mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V^\sharp), L) \cong \mathrm{MC}(V \otimes L)$.

For $k \geq 2$, let ℓ_k denote, as in (3), the k-th bracket induced on $V \otimes L$. Then, the proof of the following analogue of Lemma 3.3 is also straightforward.

Lemma 3.6. *For each* $T \in \mathscr{P}\mathscr{T}_k$,

$$\widetilde{\ell}_T(a_1 \otimes x_1, \dots, a_k \otimes x_k) = m_T(a_1 \otimes \cdots \otimes a_k) \otimes \gamma \circ Q_T(x_1 \otimes \cdots \otimes x_k).$$

$\square$

Proof of Theorem 3.5. We have to prove that $f \in \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}(V), L)$ if and only if $fs^{-1}_{|V} = \Psi(z)$ with $z \in \mathrm{MC}(V \otimes L)$.

For it, let $z_i \in V \otimes L$ and let $\Psi(z_i) = g_i \in \mathrm{Hom}(V^\sharp, L)$, $i = 1, \dots, k$ with $k \geq 2$. Then, it is a mere computation using Lemma 3.6 to show that, for each $T \in \mathscr{P}\mathscr{T}_k$,

$$\Psi\big(\widetilde{\ell}_T(z_1, \dots, z_k)\big) = \gamma \circ Q_T \circ (g_1 \otimes \cdots \otimes g_k) \circ m_T^\sharp.$$

In particular, choosing $z_i = z$ for all i, with $\Psi(z) = g$, and applying formula (3), we get:

$$\Psi\big(\frac{1}{k!}\ell_k(z, \dots, z)\big) = \sum_{T \in \mathscr{T}_k} \frac{\gamma \circ Q_T \circ g^{\otimes k} \circ m_T^\sharp}{|\operatorname{Aut} T|}, \quad k \geq 2,$$

while, for $k = 1$, $\Psi(\ell_1 z)$ is the usual differential $\partial g - (-1)^{|g|} g d_1$ on $\mathrm{Hom}(V^\sharp, L)$.

Hence, we need to prove that given $f\colon \mathfrak{L}(V) \to L$ of degree zero and $v \in V$, the following two equations are equivalent:

$$\partial f(s^{-1}v) + \sum_{k\geq 1} f(d_k s^{-1}v) = 0,$$

$$\partial f(s^{-1}v) + f(d_1 s^{-1}v) + \sum_{k\geq 2}\sum_{T\in\mathscr{T}_k} \frac{\gamma\circ Q_T\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp}{|\operatorname{Aut} T|}(v) = 0.$$

That is,

$$\sum_{k\geq 2} f(d_k s^{-1}v) = 0 \quad \text{if and only if} \quad \sum_{k\geq 2}\sum_{T\in\mathscr{T}_k} \frac{\gamma\circ Q_T\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp}{|\operatorname{Aut} T|}(v) = 0.$$

We finish by checking that, in fact,

$$f d_k s^{-1} = -\sum_{T\in\mathscr{T}_k} \frac{\gamma\circ Q_T\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp}{|\operatorname{Aut} T|}, \quad \text{for any } k\geq 2.$$

On the one hand, $(fs^{-1})^{\otimes k}\circ m_T^\sharp$ is a map of the form (5) since $V^\sharp$ is a C_∞-coalgebra. Hence, by Theorem 2.1,

$$\sum_{T\in\mathscr{T}_k} \frac{\gamma\circ Q_T\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp}{|\operatorname{Aut} T|} = \sum_{T\in\mathscr{P}\mathscr{T}_k} \gamma\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp.$$

On the other hand, the computation in (8) replacing Δ_T by $m_T^\sharp$, provides

$$f d_k s^{-1} = -\sum_{T\in\mathscr{P}\mathscr{T}_k} \gamma\circ (fs^{-1})^{\otimes k}\circ m_T^\sharp$$

□

4 Realization of Lie algebras

For each $n\geq 1$ let Δ^n be the standard n-simplex and write in the same way the graded vector space of its simplicial chains. We denote by $a_{i_0\dots i_k}$, $0\leq i_0<\dots<i_k\leq n$, the generator of the corresponding k-simplex. Then [3, thms. 2.3 and 2.8], there is a unique (up to isomorphism) cDGL of the form $\mathfrak{L}_n = (\widehat{\mathbb{L}}(s^{-1}\Delta^n), \partial)$ such that:

(1) For each $i=0,\dots,n$, the generators $s^{-1}a_i \in s^{-1}\Delta^n_0$ corresponding to vertices are Maurer-Cartan elements, $\partial s^{-1}a_i = -\frac{1}{2}[s^{-1}a_i, s^{-1}a_i]$.

(2) The linear part ∂_1 of ∂ is precisely the differential of the simplicial chain complex $s^{-1}\Delta^n$.

Moreover [3, §3], there is a natural cosimplicial cDGL structure on $\mathfrak{L}_\bullet$ and we define the realization of any cDGL L as the simplicial set,

$$\langle L\rangle = \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}_\bullet, L).$$

On the other hand denote by $\mathscr{A}_\bullet$ the simplicial CDGA of PL-forms on the standard simplices,

$$\mathscr{A}_n = \Lambda(t_0,\dots,t_n,dt_0,\dots,dt_n)/(\textstyle\sum t_i - 1, \sum dt_i),$$

and let $C^*(\Delta^\bullet)$ be the simplicial cochain complex also on the standard simplices. Then [6, 7, 9], there is a homotopy retract

$$K_\bullet \circlearrowright \mathscr{A}_\bullet \underset{i_\bullet}{\overset{p_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet),$$

where the maps $p_\bullet$ and $i_\bullet$ are defined as follows:

Let $\alpha_{i_0\dots i_k}$ be the basis for $C^*(\Delta^n)$ defined by

$$\langle \alpha_{i_0\dots i_k}, a_{j_0\dots j_k}\rangle = \begin{cases} (-1)^{\frac{k(k-1)}{2}} & \text{if } (j_0,\dots,j_k) = (i_0,\dots,i_k),\\ 0 & \text{otherwise.}\end{cases}$$

Then, $i_n(\alpha_{i_0\dots i_k})$ is the Whitney elementary form $\omega_{i_0\dots i_k}$ defined by

$$\omega_{i_0\dots i_k} = k!\sum_{j=0}^{k}(-1)^j t_{i_j} dt_{i_0}\cdots\widehat{dt_{ij}}\cdots dt_{i_k}.$$

The map $p_n\colon \mathscr{A}_n \to C^*(\Delta^n)$ is defined by

$$p_n(\omega) = \sum_{k=0}^{n}\sum_{i_0<\cdots<i_k}\alpha_{i_0\dots i_k}\mathcal{I}_{i_0\dots i_k}(\omega),$$

with

$$\mathcal{I}_{i_0\dots i_k}(t_{i_1}^{b_1}\dots t_{i_k}^{b_k}dt_{i_1}\dots dt_{i_k}) = \frac{b_1!\cdots b_k!}{(b_1+\cdots+b_k+k)!},$$

and 0 otherwise. In particular, $\mathcal{I}_{i_0\dots i_k}(\omega_{i_0\dots i_k}) = 1$.

Theorem 1.1 induces a simplicial C_∞-coalgebra structure on $C^*(\Delta^\bullet)$ and, since this is a finite dimensional simplicial cochain complex, Remark 3.4 provides by dualizing a C_∞-algebra structure on the simplicial chain complex

$\Delta^\bullet$ and a cosimplicial cDGL of the form $\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet)$. By uniqueness [3, Thm. 2.3] it follows that $\widehat{\mathbb{L}}(s^{-1}\Delta^\bullet) = \mathfrak{L}_\bullet$

Next, let L be a cDGL and consider the simplicial homotopy retract

$$\begin{tikzcd} \mathscr{A}_\bullet \otimes L \arrow[loop left, "K_\bullet\otimes \mathrm{id}_L"] \arrow[r, shift left, "p_\bullet\otimes\mathrm{id}_L"] & C^*(\Delta^\bullet)\otimes L \arrow[l, shift left, "i_\bullet\otimes\mathrm{id}_L"] \end{tikzcd}$$

which, via Theorem 1.1, provides quasi-isomorphisms of L_∞-algebras,

$$\mathscr{A}_\bullet \otimes L \underset{I_\bullet}{\overset{P_\bullet}{\rightleftarrows}} C^*(\Delta^\bullet)\otimes L,$$

and in particular, simplicial set maps,

$$\mathrm{MC}(\mathscr{A}_\bullet \otimes L) \underset{I_\bullet}{\overset{P_\bullet}{\rightleftarrows}} \mathrm{MC}(C^*(\Delta^\bullet)\otimes L).$$

We now apply Theorem 3.5 to identify the simplicial sets

$$\mathrm{MC}(C^*(\Delta^\bullet)\otimes L) \cong \mathrm{Hom}_{\mathrm{cDGL}}(\mathfrak{L}_\bullet, L) \tag{9}$$

and therefore we have simplicial maps,

$$\mathrm{MC}(\mathscr{A}_\bullet \otimes L) \underset{I_\bullet}{\overset{P_\bullet}{\rightleftarrows}} \langle L\rangle_\bullet,$$

relating our realization with the Hinich "contents or nerve" of L [12]. The following is Theorem 0.1.

Theorem 4.1. *The maps $I_\bullet$ and $P_\bullet$ are homotopy equivalences which make $\langle L\rangle_\bullet$ a strong deformation retract of* $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$.

Proof. Theorem 1.1 provides $P_\bullet I_\bullet = \mathrm{id}_{\langle L\rangle}$. Also note that both $\langle L\rangle_\bullet$ and $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$ are Kan complexes, see [3] and [9] respectively. We then finish by showing that $I_\bullet$ is a weak homotopy equivalence.

On the one hand, both $\pi_0\langle L\rangle_\bullet$ and $\pi_0\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$ coincide with the set $\widetilde{\mathrm{MC}}(L)$ of Maurer-Cartan elements of L modulo the gauge relation, see [3, Prop. 4.4] and [9, Introduction] respectively. Now, for any $z \in \mathrm{MC}(L)$ consider the "localization" of L at z which is the cDGL

$$L^{(z)} = (L, \partial_z)/(L_{<0} \oplus M)$$

in which $\partial_z = \partial + \mathrm{ad}_z$ is the original differential on L perturbed by the adjoint on z, and M is a complement of $\ker \partial_z$ in L_0. Then [3, Prop. 4.6], the path component $\langle L\rangle_z$ of $\langle L\rangle_\bullet$ at $z \in \widetilde{\mathrm{MC}}(L)$ is precisely $\langle L^{(z)}\rangle_\bullet$. Moreover [3, Prop. 4.5], and this is valid for any non-negatively graded cDGL, for any $n \geq 1$, there is a natural group isomorphism,

$$\varphi\colon H_{n-1}(L^{(z)}) \xrightarrow{\cong} \pi_n\langle L^{(z)}\rangle_\bullet, \qquad \varphi[\alpha] = \overline{f},$$

where this is the homotopy class represented by the the cDGL morphism $f\colon \widehat{\mathbb{L}}(s^{-1}\Delta^n) \to L^{(z)}$ which takes the top generator $s^{-1}a_{0\dots n}$ to Φ, and is zero on the rest of generators. For completeness, we remark that for $n = 1$ the group structure on $H_0(L^{(z)})$ is given by the Baker-Campbell-Hausdorff formula. Composing φ with the morphism induced in homotopy groups by the isomorphism (9) we obtain a group isomorphism:

$$\psi\colon H_{n-1}(L^{(z)}) \xrightarrow{\cong} \pi_n\mathrm{MC}(C^*(\Delta^\bullet) \otimes L^{(z)}), \quad \psi[\Phi] = \overline{\alpha_{0\dots n} \otimes \Phi}.$$

On the other hand, in [1, Cor.1.3] (cf. [17, Prop. 7.20]) it is proved that the path component $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)_z$ of $\mathrm{MC}(\mathscr{A}_\bullet \otimes L)$ at $z \in \widetilde{\mathrm{MC}}(L)$ is precisely $\mathrm{MC}(\mathscr{A}_\bullet \otimes L^{(z)})$. Moreover [1, Thm. 1] (see §4 of this reference for details), for each $n \geq 1$, there is a group isomorphism $H_{n-1}(L^{(z)}) \cong \pi_n\mathrm{MC}(\mathscr{A}_\bullet \otimes L^{(z)})$ which sends the homology class $[\Phi]$ to $\overline{\omega_{0\dots n} \otimes \Phi}$. We finish by checking that, for each $n \geq 1$, the induced map at the component determined by z,

$$I_n\colon \mathrm{MC}(C^*(\Delta^n) \otimes L^{(z)}) \to \mathrm{MC}(\mathscr{A}_n \otimes L^{(z)}),$$

satisfies $I_n(\alpha_{0\dots n} \otimes \Phi) = \omega_{0\dots n} \otimes \Phi$.

As a general fact, the induced map I_n at the respective Maurer-Cartan sets takes any Maurer-Cartan element x to $\sum_k \frac{1}{k!} I_n^{(k)}(x, \dots, x)$ being $\{I_n^{(k)}\}_{k\geq 1}$ the Taylor series of the L_∞-morphism I_n. On the other hand, Theorem 1.1 gives an explicit recursive description of this series,

$$I_n^{(k)}(x_1, ..., x_k) = \sum_{i=1}^{k-1} \sum_{\sigma\in\widetilde{S}(i,k-i)} \varepsilon(\sigma) K[I_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), I_{k-i}(x_{\sigma(i+1)}, ..., x_{\sigma(k)})]$$

being K the chain homotopy and $\widetilde{S}(i, k-i)$ the shuffle permutations which fix 1.

In our particular case, as any power of $\omega_{0\dots n}$ vanishes, and $i_n(\alpha_{0\dots n}) = \omega_{0\dots n}$, it follows that

$$I_n^{(1)}(\alpha_{0\dots n} \otimes \Phi) = \omega_{0\dots n} \otimes \Phi, \quad I_n^{(k)}(\alpha_{0\dots n} \otimes \Phi, \dots, \alpha_{0\dots n} \otimes \Phi) = 0,\ k \geq 2,$$

and the proof is complete. □

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